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Toeplitz Determinants with One Fisher–Hartwig Singularity

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Let c be a function defined on the unit circle with Fourier coefficients $\{c_n\}_{n=-\infty}^{\infty}$. The Fisher–Hartwig conjecture describes the asymptotic behaviour of the deter-

for a certain class of functions c. In this paper we prove this conjecture in the case of functions with one singularity. More precisely, we consider functions of the form

$$c(e^{i\theta}) = b(e^{i\theta})\, t_\beta(e^{i(\theta-\theta_1)})\, u_\alpha(e^{i(\theta-\theta_1)}).$$

Here $t_\beta(e^{i\theta}) = \exp(i\beta(\theta-\pi))$, $0<\theta<2\pi$, is a function with a jump discontinuity, $u_\alpha(e^{i\theta}) = (2-2\cos\theta)^\alpha$ is a function which may have a zero, a pole, or a discontinuity of oscillating type, and b is a sufficiently smooth nonvanishing function with winding number equal to zero. The only restriction we impose on the parameters is that 2α is required not to be a negative integer. In the case where $\operatorname{Re}\alpha \leqslant -1/2$, i.e., where the corresponding function c is not integrable, we identify c in an appropriate way with a distribution.

1. INTRODUCTION

Let c be a distribution defined on $C^\infty(\mathbb{T})$, $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$, and let $\{c_n\}_{n=-\infty}^{\infty}$ be the sequence of the Fourier coefficients of c. We consider the $n \times n$ Toeplitz matrices

$$T_n(c) := [c_{i-j}]_{i,j=0}^{n-1} \qquad (n = 1, 2, \dots). \tag{1}$$

The distribution c is called the generating distribution. Most of the considerations of Toeplitz matrices are restricted to the case where the distribution c

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can be identified with a function in $L^1(\mathbb{T})$, although this restriction is not always necessary.[1]

In analysis and in applications, there often occurs the problem of extracting some kind of information from the sequence $\{T_n(c)\}$. For instance, one asks for the behaviour of the Toeplitz determinants

$$D_n(c) := \det T_n(c) \tag{2}$$

as n tends to infinity. There are a variety of problems, particularly in statistical physics, which are closely related to this question (see [9, 10, 13]). In 1968, M. E. Fisher and R. E. Hartwig [9] singled out a class of functions c for which the asymptotic behaviour of $D_n(c)$ is of special interest, and they formulated a general conjecture about the behaviour for functions of this class. We recall the underlying definitions and the conjecture.

The functions under consideration are of the form

$$c(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)})\, u_{\alpha_r}(e^{i(\theta-\theta_r)}), \tag{3}$$

where

$$t_\beta(e^{i\theta}) = \exp(i\beta(\theta-\pi)), \qquad 0<\theta<2\pi, \tag{4}$$

$$u_\alpha(e^{i\theta}) = (2-2\cos\theta)^\alpha, \qquad \operatorname{Re}\alpha > -1/2, \tag{5}$$

and $b\colon \mathbb{T} \to \mathbb{C}$ is a sufficiently smooth function with $b(t) \neq 0$ for all $t \in \mathbb{T}$ and with winding number equal to zero:

$$\operatorname{wind} b = 0. \tag{6}$$

The assumptions on b ensure the existence of a logarithm, $\log b$, which is again a sufficiently smooth function (the precise smoothness condition will be formulated later on). We denote the nth Fourier coefficient of $\log b$ by $[\log b]_n$ and introduce the constants

[1] A distribution on $C^\infty(\mathbb{T})$ is a linear continuous functional on the space $C^\infty(\mathbb{T})$ of all infinitely differentiable functions on $\mathbb{T}$. A function $f \in L^1(\mathbb{T})$ is usually identified with the distribution T_f, where

$$T_f(g) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, g(e^{i\theta})\, d\theta, \qquad g \in C^\infty(\mathbb{T}).$$

The Fourier coefficients of an arbitrary distribution T are defined by $[T]_n = T(\chi_{-n})$, where $\chi_{-n}(t) = t^{-n}$, $t \in \mathbb{T}$. This definition ensures that for $f \in L^1(\mathbb{T})$ the Fourier coefficients of T_f coincide with those of f.

$$G[b] = \exp[\log b]_0, \tag{7}$$

$$E[b] = \exp\left(\sum_{n=1}^{\infty} n\,[\log b]_n\,[\log b]_{-n} \right). \tag{8}$$

Since $\log b$ is sufficiently smooth, the functions

$$b_+(t) = \exp\left(\sum_{n=1}^{\infty} t^n [\log b]_n \right), \qquad t \in \mathbb{T}, \tag{9}$$

$$b_-(t) = \exp\left(\sum_{n=1}^{\infty} t^{-n} [\log b]_{-n} \right), \qquad t \in \mathbb{T}, \tag{10}$$

are well-defined and smooth. Moreover, b_+ and b_- can be extended analytically to $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, respectively, and $b_+(0) = b_-(\infty) = 1$. The representation

$$b(t) = b_-(t)\, G[b]\, b_+(t), \qquad t \in \mathbb{T}, \tag{11}$$

is called the Wiener–Hopf factorization of b.

The Fisher–Hartwig conjecture in its original form asserts that

$$D_n(c) = G[b]^n\, n^{\Omega} E + o(G[b]^n\, n^{\Omega}) \tag{12}$$

as $n \to \infty$, where

$$\Omega = \sum_{r=1}^{R} (\alpha_r^2 - \beta_r^2), \tag{13}$$

$$\begin{aligned} E = E[b] &\prod_{r=1}^{R} b_+(e^{i\theta_r})^{-\alpha_r + \beta_r}\, b_-(e^{i\theta_r})^{-\alpha_r - \beta_r} \\ &\times \prod_{1 \leqslant s \neq r \leqslant R} (1 - e^{i(\theta_s - \theta_r)})^{-(\alpha_r + \beta_r)(\alpha_s - \beta_s)} \\ &\times \prod_{r=1}^{R} \frac{G(1 + \alpha_r + \beta_r)\, G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)}. \end{aligned} \tag{14}$$

Here $G(\cdot)$ stands for the Barnes G-function [16], which is an entire analytic function defined by

$$G(1+z) = (2\pi)^{z/2}\, e^{-(z+1)z/2 - \gamma_E z^2/2} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n e^{-z + z^2/2n} \right\}, \tag{15}$$

with γ_E being Euler's constant.

Note that if $\alpha_r = \beta_r = 0$ for all $1 \leqslant r \leqslant R$ then $c(t) = b(t)$ and

$$\lim_{n \to \infty} \frac{D_n(b)}{G[b]^n} = E[b].$$

This is precisely the strong Szegö limit theorem [15], which has been the subject of numerous investigations. Some historical remarks on this topic can be found in Chap. 10 of [8].

Functions of the form (3) can be very complicated. Clearly, $(2-2\cos(\theta-\theta_r))^{\alpha_r}$ has a zero at θ_r if $\operatorname{Re}\alpha_r > 0$, a pole at θ_r if $\operatorname{Re}\alpha_r < 0$ and a discontinuity of oscillating type at θ_r if $\operatorname{Re}\alpha_r = 0$ but $\operatorname{Im}\alpha_r \neq 0$. The factor $\exp(i\beta_r(\theta-\theta_r-\pi))$ is a function with a jump at θ_r. The one-sided limits are equal to $\exp(-i\pi\beta_r)$ and $\exp(i\pi\beta_r)$ as $\theta \to \theta_r + 0$ and $\theta \to \theta_r - 0$, respectively.

The Fisher–Hartwig conjecture has been confirmed in some particular cases. In 1973, H. Widom [17] proved that the conjecture is true if $\operatorname{Re}\alpha_r > -1/2$ and $\beta_r = 0$ for all r. In 1978, E. Basor [1] extended the result to the case where $\operatorname{Re}\alpha_r > -1/2$ and $\operatorname{Re}\beta_r = 0$ for all r. In 1985, A. Böttcher and one of the authors [6] confirmed the conjecture if $|\operatorname{Re}\alpha_r| < 1/2$ and $|\operatorname{Re}\beta_r| < 1/2$ for all r. The latter result is, in a certain sense, the best thing one can prove: It is known that the conjecture need not be true if the function c has at least two singularities of a "size" greater than or equal to 1/2. This was the reason to formulate an extended version of the Fisher–Hartwig conjecture (see [2–5]).

In contrast, much more can be said if c has only one singularity ($R = 1$). We assume without loss of generality that $\theta_1 = 0$, and we write $\alpha = \alpha_1$, $\beta = \beta_1$ for brevity. The conjecture was shown to be valid if $\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}(\alpha-\beta) > -1$ and $\operatorname{Re}(\alpha+\beta) > -1$ (see [7]). R. Libby [12] confirmed the conjecture in the case $\alpha = 0$ and $|\operatorname{Re}\beta| < 5/2$. Moreover, all is known if $b(t) = 1$, i.e., if $c(e^{i\theta}) = u_\alpha(e^{i\theta})\, t_\beta(e^{i\theta})$. In [6], the corresponding Toeplitz determinant was explicitly calculated. It turned out that

$$D_n(c) = \frac{G(1+\alpha+\beta)\, G(1+\alpha-\beta)}{G(1+2\alpha)} \cdot \frac{G(1+n)\, G(1+n+2\alpha)}{G(1+n+\alpha+\beta)\, G(1+n+\alpha-\beta)} \tag{16}$$

for all $n \geqslant 1$ if $\operatorname{Re}\alpha > -1/2$ and neither $\alpha+\beta$ nor $\alpha-\beta$ is a negative integer, whereas $D_n(c) = 0$ for all $n \geqslant 1$ if $\operatorname{Re}\alpha > -1/2$ and either $\alpha+\beta$ or $\alpha-\beta$ is a negative integer. Notice that the second factor in (16) behaves asymptotically as $n^{\alpha^2-\beta^2}$.

It is the aim of the present paper to prove the Fisher–Hartwig conjecture in the case of one singularity in full generality. Doing this, we will drop the condition $\operatorname{Re}\alpha > -1/2$. The problem, which occurs here for a moment, is how to associate a distribution to the function

$$c(t) = b(t)\, t_\beta(t)\, u_\alpha(t), \qquad t \in \mathbb{T}. \tag{17}$$

In the case where $\operatorname{Re}\alpha > -1/2$, this is clear since $c \in L^1(\mathbb{T})$. In the next section, we will explain how this can be done in general. It turns out that this is possible in a reasonable way if only $2\alpha \notin \mathbb{Z}^- := \{-1, -2, -3, ...\}$. This condition will be the only one we impose on the parameters α and β. After the distribution has been determined, its Fourier coefficients are well-defined and, consequently, so is $D_n(c)$.

The result which we obtain contains also an estimate of the speed of convergence in (12), and we prove that the convergence is uniform with respect to the parameters α and β (on compact sets).

Instead of considering t_β and u_α, it is more convenient to work with the functions

$$\eta_\gamma(t) = (1-t)^\gamma, \qquad |t| \leqslant 1, \qquad t \neq 1,$$

$$\xi_\delta(t) = (1-1/t)^\delta, \qquad |t| \geqslant 1, \qquad t \neq 1.$$

Here we choose the branches of the analytic (in t) functions for which $\eta_\gamma(0) = \xi_\delta(\infty) = 1$. It is easy to verify that

$$t_\beta(e^{i\theta})\, u_\alpha(e^{i\theta}) = \xi_\delta(e^{i\theta})\, \eta_\gamma(e^{i\theta})$$

for all $\theta \in (0, 2\pi)$ if $\gamma = \alpha + \beta$ and $\delta = \alpha - \beta$. Therefore, we will henceforth consider the parameters γ and δ instead of α and β.

2. PRELIMINARIES AND THE MAIN RESULT

In this section, we introduce classes of smooth functions on and a class of distributions.

Let $\mathbb{N} := \{0, 1, 2, ...\}$, and let ℓ^p_μ, $1 \leqslant p < \infty$ and $\mu \in \mathbb{R}$, stand for the weighted space of sequences $\{x_n\}_{n=0}^\infty$ for which

$$\|\{x_n\}_{n=0}^\infty\|_{\ell^p_\mu} = \left(\sum_{n=0}^{\infty} (1+n)^{\mu p} \, |x_n|^p \right)^{1/p} < \infty. \tag{21}$$

By $F\ell^{p,p}_{\varrho_1,\varrho_2}$ we denote the set of all functions $a \in L^1(\mathbb{T})$ whose Fourier coefficients satisfy $\{a_n\}_{n=0}^\infty \in \ell^p_{\varrho_1}$ and $\{a_{-n}\}_{n=0}^\infty \in \ell^p_{\varrho_2}$. The norm of a function $a \in F\ell^{p,p}_{\varrho_1,\varrho_2}$ is defined by

$$\|a\|_{F\ell^{p,p}_{\varrho_1,\varrho_2}} = \left(\sum_{n=0}^{\infty} (1+n)^{\varrho_1 p} \, |a_n|^p + \sum_{n=1}^{\infty} (1+n)^{\varrho_2 p} \, |a_{-n}|^p \right)^{1/p}.$$

It is known that $F\ell^{1,1}_{\varrho,\varrho}$ (if $\varrho \geqslant 0$) and $F\ell^{2,2}_{\varrho_1,\varrho_2}$ (if $\varrho_1, \varrho_2 > 1/2$) are Banach algebras of continuous functions on $\mathbb{T}$, where the algebraic operations are defined pointwise (see Sect. 6.54 and 6.55 of [8]).

Let $b \in F\ell^{2,2}_{\varrho_1,\varrho_2}$ with $\varrho_1, \varrho_2 > 1/2$. Suppose in addition that b is invertible and has a winding number equal to zero. Then b possesses a logarithm $\log b \in F\ell^{2,2}_{\varrho_1,\varrho_2}$. Consequently, the constants $G[b]$ and $E[b]$ are well defined, and the functions b_+ and b_- as well as their inverses b_+^{-1} and b_-^{-1} are contained in $F\ell^{2,2}_{\varrho_1,\varrho_2}$.

Let $\Gamma(\cdot)$ denote the gamma function. As is known, both $\Gamma(1+z)$ and $1/\Gamma(1+z)$ are analytic on $\mathbb{C}\setminus\mathbb{Z}^-$. Moreover, $1/\Gamma(1+z)$ can be analytically continued to all of $\mathbb{C}$ by stipulating $1/\Gamma(1+z) := 0$ for $z \in \mathbb{Z}^-$. Therefore, we will henceforth consider $1/\Gamma(1+z)$ as an entire analytic function. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, let $(\alpha)_n$ and $\mu_n^{(\alpha)}$ denote

$$(\alpha)_n := \alpha\,(\alpha+1)\cdot \cdots \cdot (\alpha+n-1), \tag{23}$$

$$\mu_n^{(\alpha)} := \frac{(1+\alpha)_n}{n!}. \tag{24}$$

Note that $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ if $\alpha+n-1 \notin \mathbb{Z}^-$.

Lemma 2.1. *For each compact subset K of $\mathbb{C}$, there is a constant C such that for all $\alpha \in K$ and all $n \in \mathbb{N}$ the following two estimates hold:*

$$\text{(a)} \qquad |\mu_n^{(\alpha)}| \leqslant C(1+n)^{\operatorname{Re}\alpha},$$

$$\text{(b)} \qquad \left|\frac{n!}{\Gamma(1+\alpha+n)}\right| \leqslant C(1+n)^{-\operatorname{Re}\alpha}.$$

Proof. Using Euler's formula for the gamma function [16], it is easy to verify that

$$\frac{1}{\Gamma(1+\alpha)} = \lim_{n\to\infty} \frac{(1+\alpha)_n}{n!}(1+n)^{-\alpha},$$

where the convergence is uniform on compact subsets K of $\mathbb{C}$. From this we deduce by a simple computation that

$$1 = \lim_{n\to\infty} \frac{n!}{\Gamma(1+\alpha+n)}(1+n)^{\alpha}$$

uniformly on compact subsets K of $\mathbb{C}\setminus\mathbb{Z}^-$. Observing that the functions under the limit are entire analytic functions in α, and applying the maximum modulus principle, we see that the convergence is uniform on each compact subset K of $\mathbb{C}$. ∎

Our next objective is to define a class of distributions $a_{\gamma,\delta}$, where γ and δ are complex numbers with $\gamma+\delta \notin \mathbb{Z}^-$. We want to choose the distributions $a_{\gamma,\delta}$ in such a way that they can be regarded as associated to the functions $\xi_\delta \eta_\gamma$. The definition is based on their Fourier coefficients, and we will therefore use the following result (see [11]).

PROPOSITION 2.2. *Let $\{c_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers for which there are constants C and ϱ such that $|c_n| \leqslant C(1+|n|)^{\varrho}$ for all $n \in \mathbb{Z}$. Then there is a uniquely determined distribution c on $C^\infty(\mathbb{T})$ whose Fourier coefficients coincide with $\{c_n\}_{n=-\infty}^{\infty}$.*

Consider the functions $\xi_\delta \eta_\gamma$. If $\mathrm{Re}(\gamma+\delta) > -1$, then $\xi_\delta \eta_\gamma \in L^1(\mathbb{T})$, and the Fourier coefficients can be calculated explicitly (see Sect. 6.18 of [8]): the nth Fourier coefficient of $\xi_\delta \eta_\gamma$ is equal to

$$(-1)^n \frac{\Gamma(1+\gamma+\delta)}{\Gamma(1+\gamma-n)\,\Gamma(1+\delta+n)}. \tag{25}$$

In the case where $\gamma - n \in \mathbb{Z}^-$ or $\delta + n \in \mathbb{Z}^-$ this expression vanishes. For each fixed $n \in \mathbb{Z}$, the expression (25) represents an analytic function in γ and δ except for $\gamma+\delta \in \mathbb{Z}^-$. Hence (25) makes sense not only for $\mathrm{Re}(\gamma+\delta) > -1$, but even if $\gamma+\delta \notin \mathbb{Z}^-$. We denote it by $[a_{\gamma,\delta}]_n$. Now we examine the behaviour of $[a_{\gamma,\delta}]_n$ for $|n|$ large, i.e., we show that $[a_{\gamma,\delta}]_n$ is of at most polynomial growth.

LEMMA 2.3. *For each compact subset K of $\{(\gamma,\delta) \in \mathbb{C} \times \mathbb{C} : \gamma+\delta \notin \mathbb{Z}^-\}$ there is a constant C such that for all $(\gamma,\delta) \in K$ and all $n \in \mathbb{Z}$*

$$|[a_{\gamma,\delta}]_n| \leqslant C(1+|n|)^{-1-\mathrm{Re}(\gamma+\delta)}. \tag{26}$$

Proof. For $n \geqslant 0$, we have

$$[a_{\gamma,\delta}]_n = \frac{\Gamma(1+\gamma+\delta)}{\Gamma(1+\gamma)} \cdot \frac{(-\gamma)_n}{n!} \cdot \frac{n!}{\Gamma(1+\delta+n)},$$

and the assertion follows from Lemma 2.1. For $n \leqslant 0$, the argumentation is similar (we can interchange γ and δ). ∎

Inequality (26) in combination with Proposition 2.2 implies that there is a uniquely determined distribution $a_{\gamma,\delta}$ with Fourier coefficients $\{[a_{\gamma,\delta}]_n\}$. Clearly, if $\mathrm{Re}(\gamma+\delta) > -1$, this distribution can be identified with the function $\xi_\delta \eta_\gamma$ as it is customary.

Next we consider the question of how to define the product of a smooth function b with the distribution $a_{\gamma,\delta}$. Again, the definition is based on the Fourier coefficients.

PROPOSITION 2.4. *Let* $\varrho \geqslant 0$.

(a) *If* $b \in F\ell^{1,1}_{\varrho,\varrho}$, *then for each (fixed)* $n \in \mathbb{Z}$ *the series* $\sum_{k=-\infty}^{\infty} b_k [a_{\gamma,\delta}]_{n-k}$ *converges absolutely and uniformly on compact subsets* K *of*

$$A_\varrho := \{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C} : \gamma + \delta \notin \mathbb{Z}^-, \operatorname{Re}(\gamma + \delta) \geqslant -1 - \varrho\}$$

to a function $[ba_{\gamma,\delta}]_n$, *which is analytic in* γ *and* δ *on the interior of* A_ϱ.

(b) *If* $b, c \in F\ell^{1,1}_{\varrho,\varrho}$, *then for all* $n \in \mathbb{Z}$ *and all* $(\gamma, \delta) \in A_\varrho$ *we have*

$$[(bc)\, a_{\gamma,\delta}]_n = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_k [a_{\gamma,\delta}]_{n-k-l}\, c_l. \tag{27}$$

Proof. (a) By Lemma 2.3, we have for all $(\gamma, \delta) \in K$ and all $n \in \mathbb{Z}$

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |b_k [a_{\gamma,\delta}]_{n-k}| &\leqslant C \sum_{k=-\infty}^{\infty} |b_k|\, (1 + |n-k|)^\varrho \\ &\leqslant C(1 + |n|)^\varrho \sum_{k=-\infty}^{\infty} |b_k|\, (1 + |k|)^\varrho \\ &= C(1 + |n|)^\varrho\, \|b\|_{F\ell^{1,1}_{\varrho,\varrho}}, \end{aligned} \tag{28}$$

which proves the assertion.

(b) Since for all $n \in \mathbb{Z}$

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |b_k [a_{\gamma,\delta}]_{n-k-l}\, c_l| \\ &\quad \leqslant C \sum_{k,l} |b_k|\, |c_l|\, (1 + |n-k-l|)^\varrho \\ &\quad \leqslant C(1 + |n|)^\varrho \sum_{k,l}^{\infty} |b_k|\, |c_l|\, (1 + |k|)^\varrho\, (1 + |l|)^\varrho \\ &\quad = C(1 + |n|)^\varrho\, \|b\|_{F\ell^{1,1}_{\varrho,\varrho}}\, \|c\|_{F\ell^{1,1}_{\varrho,\varrho}}, \end{aligned}$$

the series on the right hand side of (27) converges absolutely, and the order of summation can be changed:

$$\sum_k \sum_l b_k [a_{\gamma,\delta}]_{n-k-l}\, c_l = \sum_{k'} \sum_{l'} [a_{\gamma,\delta}]_{n-k'}\, c_{k'-l'}\, b_{l'} = \sum_{k'} [a_{\gamma,\delta}]_{n-k'}\, [bc]_{k'}.$$

This is the assertion. ∎

Using (28), we see that also the numbers $[ba_{\gamma,\delta}]_n$ increase at most polynomially. By Proposition 2.2, they are the Fourier coefficients of a distribution $ba_{\gamma,\delta}$, which is by definition the product of b and $a_{\gamma,\delta}$. Again, if $\mathrm{Re}(\gamma+\delta)>-1$, then $ba_{\gamma,\delta}$ can be identified with $b\xi_\delta\eta_\gamma$ as it is customary.

Instead of characterizing the smoothness of b in terms of $F\ell^{1,1}_{\varrho,\varrho}$, we want to express it in terms of $F\ell^{2,2}_{\varrho_1,\varrho_2}$. It is known that $F\ell^{2,2}_{\varrho_1,\varrho_2}$ is continuously embedded in the space $F\ell^{1,1}_{\varrho,\varrho}$ if $\min\{\varrho_1,\varrho_2\}>1/2+\varrho$. Using this, we can restate the assertions of Proposition 2.4 as follows: let $\varrho_1,\varrho_2>1/2$ and

$$A_{\varrho_1,\varrho_2}:=\{(\gamma,\delta)\in\mathbb{C}\times\mathbb{C}:\gamma+\delta\notin\mathbb{Z}^-,\ \mathrm{Re}(\gamma+\delta)>-1/2-\min\{\varrho_1,\varrho_2\}\}.$$

Then, for $b\in F\ell^{2,2}_{\varrho_1,\varrho_2}$ and $(\gamma,\delta)\in A_{\varrho_1,\varrho_2}$, the distribution $ba_{\gamma,\delta}$ is well-defined by means of its Fourier coefficients,

$$[ba_{\gamma,\delta}]_n=\sum_{k=-\infty}^{\infty}b_k[a_{\gamma,\delta}]_{n-k}, \tag{29}$$

which in turn are analytic functions in γ and δ on A_{ϱ_1,ϱ_2}. Moreover, (27) holds if in addition $c\in F\ell^{2,2}_{\varrho_1,\varrho_2}$.

Now, after having defined the distributions $a_{\gamma,\delta}$ and $ba_{\gamma,\delta}$ which are associated to the functions $\xi_\delta\eta_\gamma$ and $b\xi_\delta\eta_\gamma$, respectively, we will no longer use the notation $a_{\gamma,\delta}$ and $ba_{\gamma,\delta}$ for the distributions, but we will denote them by $\xi_\delta\eta_\gamma$ and $b\xi_\delta\eta_\gamma$, too. This change in notation should emphasize once more that we identify the distributions with these functions in the above described way.

Now we are prepared to formulate the main result. The rest of this paper is devoted to its proof.

THEOREM 2.5 (Main Result). *Let $\varepsilon_1,\varepsilon_2>0$ and $\varepsilon=\varepsilon_1+\varepsilon_2<1$. Suppose further that $\min\{\varrho_1,\varrho_2\}>3/2-\varepsilon$ and that $b\in F\ell^{2,2}_{\varrho_1,\varrho_2}$ is an invertible function with winding number equal to zero. Then for each compact subset K of*

$$U=\{(\gamma,\delta)\in\mathbb{C}\times\mathbb{C}:\gamma+\delta\notin\mathbb{Z}^-,\ \varrho_1>c_1(\gamma,\delta),\ \varrho_2>c_2(\gamma,\delta),\ \varrho_1+\varrho_2>c_3(\gamma,\delta)\},$$

where

$$\begin{aligned}c_1(\gamma,\delta)=\max\{&1/2-\varepsilon_1+\mathrm{Re}\,\delta,\ 1/2-\varepsilon_1+\mathrm{Re}(\delta-\gamma),\ -\mathrm{Re}(\gamma+\delta)-1/2,\\&1/2-\varepsilon_2-\mathrm{Re}\,\delta,\ 3/2-\varepsilon-\mathrm{Re}\,\gamma,\ 1/2-\varepsilon_1-2\,\mathrm{Re}\,\gamma\},\\c_2(\gamma,\delta)=\max\{&1/2-\varepsilon_2+\mathrm{Re}\,\gamma,\ 1/2-\varepsilon_2+\mathrm{Re}(\gamma-\delta),\ -\mathrm{Re}(\gamma+\delta)-1/2,\\&1/2-\varepsilon_1-\mathrm{Re}\,\gamma,\ 3/2-\varepsilon-\mathrm{Re}\,\delta,\ 1/2-\varepsilon_2-2\,\mathrm{Re}\,\delta\},\\c_3(\gamma,\delta)=\max\{&2-\varepsilon+\mathrm{Re}(\gamma+\delta),\ 2-\varepsilon+\mathrm{Re}\,\gamma,\ 2-\varepsilon+\mathrm{Re}\,\delta\},\end{aligned}$$

there is a constant $C = C(K)$ *such that for all* $(\gamma, \delta) \in K$ *and all* $n \in \mathbb{N}$

$$\left| \frac{D_n(b\xi_\delta \eta_\gamma)}{G[b]^n \,(1+n)^{\gamma\delta}} - E \right| \leqslant \frac{C}{(1+n)^{1-\varepsilon}},$$

where E *is the constant*

$$E = E[b]\, b_+(1)^{-\delta}\, b_-(1)^{-\gamma}\, \frac{G(1+\gamma)\, G(1+\delta)}{G(1+\gamma+\delta)}, \tag{31}$$

and $E[b]$, $G[b]$, b_+, *and* b_- *are defined as in Section* 1.

Let us make some comments on this theorem. If we suppose that $b \in C^\infty(\mathbb{T})$, then

$$\frac{D_n(b\xi_\delta \eta_\gamma)}{G[b]^n \, n^{\gamma\delta}} = E + O(n^{-1+\varepsilon})$$

for all complex numbers γ and δ with $\gamma + \delta \notin \mathbb{Z}^-$, where the convergence is uniform on compact subsets of $\{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C} \colon \gamma + \delta \notin \mathbb{Z}^-\}$, and ε can be chosen as small as desired. In this sense, Theorem 2.5 completely confirms the Fisher–Hartwig conjecture for one singularity. Notice, however, that the case where b is merely a piecewise C^∞-function, which is of interest in some applications (see [3]), is actually not covered by the theorem (at most for small values of $\operatorname{Re} \gamma$ and $\operatorname{Re} \delta$). We believe that convergence can be shown also in this case, and we hope to come back to this point in a further publication.

Moreover, observe that the constant E vanishes if and only if $\gamma \in \mathbb{Z}^-$ or $\delta \in \mathbb{Z}^-$. In this case, we conjecture that even

$$D_n(b\xi_\delta \eta_\gamma) = o(G[b]^n \, n^\omega)$$

for all $\omega \in \mathbb{R}$, provided that $b \in C^\infty(\mathbb{T})$. For the particular case where $-\delta = \gamma \in \mathbb{Z}^-$ or $-\gamma = \delta \in \mathbb{Z}^-$, i.e. where c is a continuous and invertible function on $\mathbb{T}$ with a nonvanishing winding number, the reader is referred to Sect. 10.43 and 10.44 of [8], and to [18].

3. THE PURE FISHER–HARTWIG SINGULARITY

In what follows, we establish some formulas for Toeplitz and Hankel operators generated by $\xi_\delta \eta_\gamma$. These formulas can be regarded as the main tool for the proof of Theorem 2.5.

Let c be a distribution with Fourier coefficients $\{c_n\}_{n=-\infty}^{\infty}$. The Toeplitz operator generated by c is the infinite matrix

$$T(c) := [c_{i-j}]_{i,j=0}^{\infty}, \tag{32}$$

and the Hankel operator generated by c is the infinite matrix

$$H(c) := [c_{1+i+j}]_{i,j=0}^{\infty}. \tag{33}$$

Later, we let these infinite matrices act between appropriately chosen spaces ℓ^2_μ such that they become linear bounded operators. In this section, however, we still regard $T(c)$ and $H(c)$ exclusively as infinite matrices. Moreover, we need the projections P_n,

$$P_n\colon (x_k)_{k=0}^{\infty} \mapsto (y_k)_{k=0}^{\infty}, \qquad y_k = \begin{cases} x_k & \text{if } \ 0 \leqslant k \leqslant n-1 \\ 0 & \text{if } \ k \geqslant n, \end{cases} \tag{34}$$

and $Q_n = I - P_n$, where I is the identity matrix. The finite matrix $T_n(c)$ can obviously be identified with $P_n T(c) P_n$.

The Fourier coefficients of the functions η_γ and ξ_δ ($\operatorname{Re}\gamma > -1$, $\operatorname{Re}\delta > -1$) are just

$$[\eta_\gamma]_n = \begin{cases} 0 & \text{if } \ n < 0 \\ (-1)^n \dbinom{\gamma}{n} = \mu_n^{(-1-\gamma)} & \text{if } \ n \geqslant 0, \end{cases} \tag{35}$$

$$[\xi_\delta]_n = \begin{cases} (-1)^n \dbinom{\delta}{-n} = \mu_{-n}^{(-1-\delta)} & \text{if } \ n \leqslant 0 \\ 0 & \text{if } \ n > 0. \end{cases} \tag{36}$$

For fixed n, these values are entire analytic functions in γ and δ, respectively, and $[\eta_\gamma]_n$ and $[\xi_\delta]_n$ increase at most polynomially as $n \to \infty$ for all $\gamma, \delta \in \mathbb{C}$ (Lemma 2.1(a)). Consequently, there exist distributions (Proposition 2.2), which we also denote by η_γ and ξ_δ, such that their Fourier coefficients coincide with (35) and (36), respectively.

For $\alpha \in \mathbb{C}$, we introduce the infinite diagonal matrix M_α and the $n \times n$ diagonal matrix $M_{\alpha,n}$:

$$M_\alpha := \operatorname{diag}(\mu_0^{(\alpha)}, \mu_1^{(\alpha)}, \mu_2^{(\alpha)}, \ldots), \tag{37}$$

$$M_{\alpha,n} := \operatorname{diag}(\mu_0^{(\alpha)}, \mu_1^{(\alpha)}, \ldots, \mu_{n-1}^{(\alpha)}). \tag{38}$$

Obviously, M_α and $M_{\alpha,n}$ are invertible if $\alpha \notin \mathbb{Z}^-$. Finally, for $\gamma, \delta \in \mathbb{C} \backslash \mathbb{Z}^-$, we abbreviate $\Gamma_{\gamma,\delta} := \Gamma(1+\gamma)\,\Gamma(1+\delta)/\Gamma(1+\gamma+\delta)$.

THEOREM 3.1. *For $\gamma, \delta \in \mathbb{C} \backslash \mathbb{Z}^-$ with $\gamma + \delta \notin \mathbb{Z}^-$ we have*

$$T(\xi_\delta \eta_\gamma) = \Gamma_{\gamma,\delta}^{-1} M_\delta^{-1} T(\eta_\gamma)\, M_{\gamma+\delta} T(\xi_\delta)\, M_\gamma^{-1}, \tag{39}$$

$$T_n(\xi_\delta \eta_\gamma) = \Gamma_{\gamma,\delta}^{-1} M_{\delta,n}^{-1} T_n(\eta_\gamma)\, M_{\gamma+\delta,n} T_n(\xi_\delta)\, M_{\gamma,n}^{-1}. \tag{40}$$

Proof. Since $T(\eta_\gamma)$ and $T(\xi_\delta)$ are lower (resp. upper) triangular matrices, the infinite matrix on the right hand side of (39) is well defined, and their entries are analytic in γ and δ. A proof of (39) can be found in [8], Theorem 6.20. The condition $\mathrm{Re}(\gamma+\delta) > -1$, which was required there, can be removed (e.g., by the analyticity of the entries). Finally, (40) follows from (39) by observing that $P_n T(\eta_\gamma) = T_n(\eta_\gamma)$ and $T(\xi_\delta) P_n = T_n(\xi_\delta)$. ∎

Formula (40) has two important implications. The first one is an explicit expression for the determinant $D_n(\xi_\delta \eta_\gamma)$:

$$\begin{aligned} D_n(\xi_\delta \eta_\gamma) &= \prod_{i=0}^{n-1} \frac{\Gamma(1+\gamma+\delta+i)\, \Gamma(1+i)}{\Gamma(1+\gamma+i)\, \Gamma(1+\delta+i)} \\ &= \frac{G(1+\gamma)\, G(1+\delta)}{G(1+\gamma+\delta)} \cdot \frac{G(1+\gamma+\delta+n)\, G(1+n)}{G(1+\gamma+n)\, G(1+\delta+n)}. \end{aligned} \tag{41}$$

Here we have used $D_n(\eta_\gamma) = D_n(\xi_\delta) = 1$ as well as the relations $G(1+z) = \Gamma(z)\, G(z)$ and $G(1) = 1$ (see [16]). The determinant vanishes if and only if $\gamma \in \mathbb{Z}^-$ or $\delta \in \mathbb{Z}^-$. The asymptotic behaviour of $D_n(\xi_\delta \eta_\gamma)$ can now be characterized as follows:

COROLLARY 3.2. *Let K be a compact subset of $\{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C} : \gamma + \delta \notin \mathbb{Z}^-\}$. Then there is a constant C such that for all $(\gamma, \delta) \in K$ and all $n \in \mathbb{N}$*

$$\left| \frac{D_n(\xi_\delta \eta_\gamma)}{(1+n)^{\gamma\delta}} - \frac{G(1+\gamma)\, G(1+\delta)}{G(1+\gamma+\delta)} \right| \leqslant \frac{C}{1+n}. \tag{42}$$

Proof. From the definition (15) of the Barnes G-function and from the definition of Euler's constant we obtain

$$G(1+z) = (2\pi)^{z/2}\, e^{-(z+1)\,z/2} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^k \left(1+\frac{1}{k}\right)^{z^2/2} e^{-z} \right\}.$$

Further, we recall Euler's formula for the gamma function [16]:

$$\Gamma(1+z) = \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-1} \left(1+\frac{1}{k}\right)^{z} \right\}.$$

Using recurrence formulas and $G(1)=\Gamma(1)=1$, we conclude that

$$\frac{G(1+z+n)}{G(1+n)}=G(1+z)\prod_{k=1}^{n}\frac{\Gamma(z+k)}{\Gamma(k)}$$
$$=G(1+z)\,\Gamma(1+z)^{n}\prod_{k=1}^{\infty}\left(\frac{k+z}{k}\right)^{n-k}.$$

We introduce the functions $S_n(z)$ by

$$S_n(z)=e^{-(z-1)z/2}\prod_{k=n+1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{k}\left(1+\frac{1}{k}\right)^{z^2/2}e^{-z}\left(1+\frac{z}{k}\right)^{-n}\left(1+\frac{1}{k}\right)^{zn}\right\}.$$

Then it follows that

$$\frac{G(1+z+n)}{G(1+n)}=(2\pi)^{z/2}\,e^{-z}S_n(z)$$
$$\times\prod_{k=1}^{n}\left\{\left(1+\frac{z}{k}\right)^{k-n}\left(1+\frac{1}{k}\right)^{z^2/2+zn}e^{-z}\left(\frac{k+z}{k}\right)^{n-k}\right\}$$
$$=(2\pi)^{z/2}\,e^{-(n+1)z}(1+n)^{z^2/2+zn}\,S_n(z)$$

Considering the asymptotic expansions (first for $k\to\infty$ and then for $n\to\infty$) in

$$\frac{-(z-1)z}{2}+\sum_{k=n+1}^{\infty}\left\{k\log\left(1+\frac{z}{k}\right)+\frac{z^2}{2}\log\left(1+\frac{1}{k}\right)-z\right\}$$
$$+\sum_{k=n+1}^{\infty}n\left\{z\log\left(1+\frac{1}{k}\right)-\log\left(1+\frac{z}{k}\right)\right\}$$

one can show straightforwardly that $\log S_n(z)=O(1/n)$. Thus, with certain positive real numbers a_n and b_n not depending on z, we have

$$G(1+z+n)=a_n b_n^z(1+n)^{z^2/2}\exp(O(1/n)),$$

which implies that (uniformly on compact subset of $\{(\gamma,\delta)\in\mathbb{C}\times\mathbb{C}:\gamma,\delta,\gamma+\delta\notin\mathbb{Z}^-\}$)

$$\frac{G(1+\gamma+\delta+n)\,G(1+n)}{G(1+\gamma+n)\,G(1+\delta+n)}=(1+n)^{\gamma\delta}\,(1+O(1/n)).$$

Since $D_n(\xi_\delta n_\gamma)$ is analytic in γ and δ, we obtain the desired assertion from (41) by help of the maximum modulus principle. ∎

The second implication of (40) is an explicit formula for the inverse of the $n\times n$ Toeplitz matrix $T_n(\xi_\delta\eta_\gamma)$. For this, observe that $T_n(\xi_\delta)^{-1}=T_n(\xi_{-\delta})$ and $T_n(\eta_\gamma)^{-1}=T_n(\eta_{-\gamma})$:

$$T_n(\xi_\delta\eta_\gamma)^{-1}=\Gamma_{\gamma,\delta}M_{\gamma,n}T_n(\xi_{-\delta})\,M^{-1}_{\gamma+\delta,n}T_n(\eta_{-\gamma})\,M_{\delta,n}.$$

For $\alpha\in\mathbb{C}\backslash\mathbb{Z}^-$, let $H(\tau_\alpha)$ be the Hankel operator defined by

$$H(\tau_\alpha):=[\tau_{\alpha,1+i+j}]_{i,j=0}^\infty,$$

where $\tau_{\alpha,1+n}:=-n!/(1+\alpha)_{1+n}$ for all $n\in\mathbb{N}$. The argument τ_α of the Hankel operator can be considered as a distribution, if we put for instance the non-positive Fourier coefficients $\tau_{\alpha,n}:=0$ for $n\leqslant 0$. We will use τ_α only in conjunction with the above Hankel operator.

THEOREM 3.3. *For $\gamma,\delta\in\mathbb{C}\backslash\mathbb{Z}^-$ with $\gamma+\delta\notin\mathbb{Z}^-$ we have*

$$\begin{aligned}H(\xi_\gamma\eta_\delta)\,M_\gamma T(\xi_{-\delta})&=\delta\Gamma^{-1}_{\gamma,\delta}M_{-\delta}H(\tau_\gamma)\,M_{\gamma+\delta},\\ T(\eta_{-\gamma})\,M_\delta\;H(\xi_\delta\eta_\gamma)&=\gamma\Gamma^{-1}_{\gamma,\delta}M_{\gamma+\delta}H(\tau_\delta)\,M_{-\gamma}.\end{aligned}\tag{45}$$

Proof. We prove only (45). Formula (46) can be obtained by taking the transposed matrix. The entries of the well-defined infinite matrix $S=H(\xi_\gamma\eta_\delta)\,M_\gamma T(\xi_{-\delta})$ are

$$S_{ij}=\sum_{k=0}^{j}[\xi_\gamma\eta_\delta]_{1+i+k}\,\mu_k^{(\gamma)}[\xi_{-\delta}]_{k-j}.$$

In order to calculate S_{ij}, we consider for each fixed i the analytic functions ($|z|<1$)

$$\sum_{k=0}^\infty z^k[\xi_{-\delta}]_{-k}=\sum_{k=0}^\infty z^k(-1)^k\binom{-\delta}{k}=(1-z)^{-\delta},$$

$$\begin{aligned}&\sum_{k=0}^\infty z^k[\xi_\gamma\eta_\delta]_{1+i+k}\,\mu_k^{(\gamma)}\\ &\quad=\sum_{k=0}^\infty z^k\frac{\Gamma(1+\gamma+\delta)}{\Gamma(\delta-i-k)\,\Gamma(2+\gamma+i+k)}(-1)^{1+i+k}\frac{(1-\gamma)_k}{k!}\\ &\quad=\frac{\Gamma(1+\gamma+\delta)}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i)}\sum_{k=0}^\infty z^k(-1)^{1+i+k}\frac{(1+\gamma)_k\,(\delta-i-k)_k}{k!\,(2+\gamma+i)_k}\\ &\quad=\frac{\Gamma(1+\gamma+\delta)(-1)^{1+i}}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i)}\sum_{k=0}^\infty z^k\frac{(1+\gamma)_k\,(1-\delta+i)_k}{k!\,(2+\gamma+i)_k}\\ &\quad=\frac{\Gamma(1+\gamma+\delta)(-1)^{1+i}}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i)}F(1-\delta+i,1+\gamma,2+\gamma+i;z),\end{aligned}$$

where $F(\cdot, \cdot, \cdot; z)$ is the hypergeometric function. From the well-known relation

$$F(A, B, C; z) = (1-z)^{C-A-B} F(C-A, C-B, C; z)$$

(see Chap. 14.4 of [16]) with $A = 1+\gamma+\delta$, $B = 1+i$, $C = 2+\gamma+i$ we conclude that

$$F(1+\gamma+\delta, 1+i, 2+\gamma+i; z) = (1-z)^{-\delta} F(1-\delta+i, 1+\gamma, 2+\gamma+i; z).$$

Hence the product of the two above considered functions is equal to the function

$$\begin{aligned} &\frac{\Gamma(1+\gamma+\delta)(-1)^{1+i}}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i)} \sum_{k=0}^{\infty} \frac{(1+\gamma+\delta)_k\,(1+i)_k}{k!\,(2+\gamma+i)_k} \\ &\quad = \sum_{k=0}^{\infty} z^k \frac{\Gamma(1+\gamma+\delta+k)(-1)^{1+i}}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i+k)} \cdot \frac{(i+k)!}{i!\,k!}. \end{aligned}$$

Comparing the coefficients in the Taylor expansion of these functions at $z=0$ yields

$$\begin{aligned} S_{ij} &= \frac{\Gamma(1+\gamma+\delta+j)(-1)^{1+i}}{\Gamma(\delta-i)\,\Gamma(2+\gamma+i+j)} \cdot \frac{(i+j)!}{i!\,j!} \\ &= \frac{\Gamma(1+\gamma+\delta)}{\Gamma(1+\gamma)\,\Gamma(1+\delta)} \cdot \frac{(-1)^{1+i}\,(\delta-i)_{1+i}}{i!} \cdot \frac{(i+j)!}{(1+\gamma)_{1+i+j}} \cdot \frac{(1+\gamma+\delta)_j}{j!} \\ &= \delta \Gamma_{\gamma,\delta}^{-1} \mu_i^{(-\delta)} \tau_{\gamma,\,1+i+j} \mu_j^{(\gamma+\delta)}. \end{aligned}$$

From this, the assertion follows immediately. ∎

Multiplying from the right (resp. left) with P_n, we obtain from (45) and (46) the formulas

$$H(\xi_\gamma \eta_\delta)\, P_n M_{\gamma,n} T_n(\xi_{-\delta}) = \delta \Gamma_{\gamma,\delta}^{-1} M_{-\delta} H(\tau_\gamma)\, M_{\gamma+\delta} P_n, \tag{47}$$

$$T_n(\eta_{-\gamma})\, M_{\delta,n} P_n H(\xi_\delta \eta_\gamma) = \gamma \Gamma_{\gamma,\delta}^{-1} P_n M_{\gamma+\delta} H(\tau_\delta)\, M_{-\gamma}. \tag{48}$$

The significance of these formulas, which are stated here for the first time, can be explained by the following circumstance: Most of the proofs of particular cases of the Fisher–Hartwig conjecture which are based on separation techniques involve the consideration of such terms as for instance $T_n(\xi_\delta \eta_\gamma)^{-1} P_n H(\xi_\delta \eta_\gamma)$. More precisely, it is shown that $H(\xi_\delta \eta_\gamma)$ is a bounded operator from $\ell^{p_1}_{\mu_1}$ into $\ell^{p_2}_{\mu_2}$ and that $T_n(\xi_\delta \eta_\gamma)^{-1}$ converges strongly on $\ell^{p_2}_{\mu_2}$, desirably to $T(\xi_\delta \eta_\gamma)^{-1}$. However, one can show that, for the values of γ and δ for which the Fisher–Hartwig conjecture has so far resisted confirmation, it is not possible to find a space $\ell^{p_2}_{\mu_2}$ with the required properties. A way out of this situation is to express $T_n(\xi_\delta \eta_\gamma)^{-1}$ by means of (43) and then to apply formula (48):

$$
\begin{aligned}
&T_n(\xi_\delta \eta_\gamma)^{-1}\, P_n H(\xi_\delta \eta_\gamma)\\
&\quad = \Gamma_{\gamma,\delta} M_{\gamma,n} T_n(\xi_{-\delta})\, M^{-1}_{\gamma+\delta,n} T_n(\eta_{-\gamma})\, M_{\delta,n} P_n H(\xi_\delta \eta_\gamma)\\
&\quad = \gamma M_{\gamma,n} T_n(\xi_{-\delta})\, M^{-1}_{\gamma+\delta,n} P_n M_{\gamma+\delta} H(\tau_\delta)\, M_{-\gamma}\\
&\quad = \gamma M_\gamma T(\xi_{-\delta})\, P_n H(\tau_\delta)\, M_{-\gamma}.
\end{aligned}
$$

This term can be shown to converge to a certain operator on appropriately chosen spaces. The proof of the Fisher–Hartwig conjecture for one sigularity, which is presented below, is based essentially on this observation, although in the calculations this fact is used rather implicitely.

4. NORM ESTIMATES FOR TOEPLITZ AND HANKEL OPERATORS

In this section we establish sufficient conditions for Toeplitz and Hankel operators, which act from a space $\ell^2_{\bar{\mu}}$ into a space $\ell^2_{\underline{\mu}}$, to be bounded or Hilbert–Schmidt. In some cases, these conditions are also necessary, whereas in the other cases they seem to be at least not far away from the necessary ones. Moreover, we show that the norm can be estimated uniformly on bounded (or compact) sets of parameters.

LEMMA 4.1. *Let $\varepsilon' > 0$, and let $M \subseteq \mathbb{R} \times \mathbb{R}$ be a bounded set. Then there is a constant C such that*

$$\sum_{k=0}^{n} (1+k)^\alpha\, (1+n-k)^\beta \leqslant C(1+n)^{\max\{\alpha,\, \beta,\, \alpha+\beta+1+\varepsilon'\}}$$

for all $n \in \mathbb{N}$ and all $(\alpha, \beta) \in M$.

Proof. Straightforward. ∎

PROPOSITION 4.2. *Let $\varepsilon' > 0$, and let $M \subseteq \mathbb{R} \times \mathbb{R}$ be a bounded set. Then there is a constant C such that the Hankel operator $H(a): \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ is Hilbert–Schmidt and*

$$\|H(a)\|_{\mathscr{C}_2(\ell^2_{\bar{\mu}},\, \ell^2_{\underline{\mu}})} \leqslant C \cdot \|\{a_{1+n}\}_{n=0}^{\infty}\|_{\ell^2_{\varrho}}$$

for all functions (distributions) a with Fourier coefficients $\{a_{1+n}\}_{n=0}^{\infty} \in \ell^2_{\varrho}$ $(\varrho \in \mathbb{R})$ *and all* $(\bar{\mu}, \underline{\mu}) \in M$ *with* $\varrho \geqslant \max\{\underline{\mu}, -\bar{\mu}, \underline{\mu} - \bar{\mu} + \frac{1}{2} + \varepsilon'\}$.

Proof. The square of the Hilbert–Schmidt norm of $H(a)$ can be estimated by

$$\begin{aligned}
&\sum_{i,k \geqslant 0} (1+k)^{2\underline{\mu}} (1+i)^{-2\bar{\mu}} |a_{1+i+k}|^2 \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (1+k)^{2\underline{\mu}} (1+n-k)^{-2\bar{\mu}} |a_{1+n}|^2 \\
&\quad \leqslant C \sum_{n=0}^{\infty} |a_{1+n}|^2 (1+n)^{\max\{2\underline{\mu}, -2\bar{\mu}, 2\underline{\mu} - 2\bar{\mu} + 1 + 2\varepsilon'\}} \\
&\qquad\qquad \text{(by Lemma 4.1)} \\
&\quad \leqslant C \sum_{n=0}^{\infty} |a_{1+n}|^2 (1+n)^{2\varrho} = C \cdot \| \{a_{1+n}\}_{n=0}^{\infty} \|^2_{\ell^2_{\varrho}}. \quad \blacksquare
\end{aligned}$$

PROPOSITION 4.3. *Let* $\varepsilon' > 0$, *let* $M \subseteq \mathbb{R} \times \mathbb{R}$ *be a bounded set, and let* $K \subseteq \mathbb{C} \backslash \mathbb{Z}^-$ *be a compact set. Then there is a constant C such that the Hankel operator* $H(\tau_\alpha) \colon \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ *is Hilbert–Schmidt and*

$$\|H(\tau_\alpha)\|_{\mathscr{C}_2(\ell^2_{\bar{\mu}}, \ell^2_{\underline{\mu}})} \leqslant C$$

for all $\alpha \in K$ *and all* $(\bar{\mu}, \underline{\mu}) \in M$ *with* $\operatorname{Re} \alpha \geqslant \varepsilon' + \max\{\underline{\mu} - \frac{1}{2}, -\bar{\mu} - \frac{1}{2}, \underline{\mu} - \bar{\mu}\}$.

Proof. We apply Proposition 4.2 with $\varepsilon'/2$ instead of ε', $a = \tau_\alpha$, and $\varrho = \operatorname{Re} \alpha + 1/2 - \varepsilon'/2$. From Lemma 2.1(b) and from (44) we conclude that

$$|\tau_{\alpha, 1+n}| \leqslant C_1 \Gamma(1+\alpha)(1+n)^{-1-\operatorname{Re} \alpha}.$$

Hence

$$\|\{\tau_{\alpha, 1+n}\}_{n=0}^{\infty}\|_{\ell^2_{\varrho}} \leqslant C_1 \Gamma(1+\alpha) \left(\sum_{n=0}^{\infty} (1+n)^{-1-\varepsilon'} \right)^{1/2},$$

which proves the assertion. $\blacksquare$

PROPOSITION 4.4. *Let* $\varepsilon' > 0$, *let* $M \subseteq \mathbb{R} \times \mathbb{R}$ *be a bounded set, and let K be a compact subset of* $\{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C} \colon \gamma + \delta \notin \mathbb{Z}^-\}$. *Then there is a constant C such that the Hankel operator* $H(\xi_\delta \eta_\gamma) \colon \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ *is Hilbert–Schmidt and*

$$\|H(\xi_\delta \eta_\gamma)\|_{\mathscr{C}_2(\ell^2_{\bar{\mu}}, \ell^2_{\underline{\mu}})} \leqslant C$$

for all $(\gamma, \delta) \in K$ *and all* $(\bar{\mu}, \underline{\mu}) \in M$ *with* $\operatorname{Re}(\gamma + \delta) \geqslant \varepsilon' + \max\{\underline{\mu} - \frac{1}{2}, -\bar{\mu} - \frac{1}{2}, \underline{\mu} - \bar{\mu}\}$.

Proof. We apply Proposition 4.2 with $\varepsilon'/2$ instead of ε', $a=\xi_\delta\eta_\gamma$ and $\varrho=\mathrm{Re}(\gamma+\delta)+1/2-\varepsilon'/2$. From Lemma 2.3 we obtain

$$|[\xi_\delta\eta_\gamma]_{1+n}|\leqslant C_1(1+n)^{-1-\mathrm{Re}(\gamma+\delta)},$$

and the assertion follows in the same manner as in the preceding proposition. ∎

PROPOSITION 4.5. *Let $\varepsilon'>0$, and let $M\subseteq\mathbb{R}\times\mathbb{R}$ be a bounded set. Then there is a constant C with the following properties*:

(a) *If $a^{(+)}$ is a function (distribution) with Fourier coefficients $\{a_n^{(+)}\}_{n=0}^{\infty}\in\ell^2_\varrho$ $(\varrho\in\mathbb{R})$ and $a_n^{(+)}=0$ if $n<0$, then the Toeplitz operator $T(a^{(+)})\colon\ell^2_{\bar\mu}\to\ell^2_{\underline{\mu}}$ is bounded and*

$$\|T(a^{(+)})\|_{\mathscr{L}(\ell^2_{\bar\mu},\,\ell^2_{\underline{\mu}})}\leqslant C\cdot\|\{a_n^{(+)}\}_{n=0}^{\infty}\|_{\ell^2_\varrho}$$

for all $(\bar\mu,\underline{\mu})\in M$ with $\varrho\geqslant\max\{\underline{\mu},\underline{\mu}-\bar\mu+\frac12+\varepsilon'\}$ and $\underline{\mu}\leqslant\bar\mu$.

(b) *If $a^{(-)}$ is a function (distribution) with Fourier coefficients $\{a_{-n}^{(-)}\}_{n=0}^{\infty}\in\ell^2_\varrho$ $(\varrho\in\mathbb{R})$ and $a_n^{(-)}=0$ if $n>0$, then the Toeplitz operator $T(a^{(-)})\colon\ell^2_{\bar\mu}\to\ell^2_{\underline{\mu}}$ is bounded and*

$$\|T(a^{(-)})\|_{\mathscr{L}(\ell^2_{\bar\mu},\,\ell^2_{\underline{\mu}})}\leqslant C\cdot\|\{a_{-n}^{(-)}\}_{n=0}^{\infty}\|_{\ell^2_\varrho}$$

for all $(\bar\mu,\underline{\mu})\in M$ with $\varrho\geqslant\max\{-\bar\mu,\underline{\mu}-\bar\mu+\frac12+\varepsilon'\}$ and $\underline{\mu}\leqslant\bar\mu$.

Proof. We prove only (a). Assertion (b) can be obtained by taking the transposed matrix. Let $x=\{x_n\}_{n=0}^{\infty}\in\ell^2_{\bar\mu}$. Then the square of the norm of $T(a^{(+)})\,x$ in $\ell^2_{\underline{\mu}}$ can be estimated by

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}a_{n-k}^{(+)}x_k\right|^2(1+n)^{2\underline{\mu}}\\
&\quad\leqslant\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}|a_{n-k}^{(+)}|^2\,|x_k|^2\,(1+k)^{2\bar\mu}\,(1+n-k)^{2\varrho}\right)\\
&\qquad\times\left(\sum_{k=0}^{n}(1+k)^{-2\bar\mu}\,(1+n-k)^{-2\varrho}\right)(1+n)^{2\underline{\mu}}\\
&\quad\leqslant A\cdot\sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_{n-k}^{(+)}|^2\,|x_k|^2\,(1+k)^{2\bar\mu}\,(1+n-k)^{2\varrho}\\
&\quad=A\cdot\|\{a_n^{(+)}\}_{n=0}^{\infty}\|^2_{\ell^2_\varrho}\cdot\|x\|^2_{\ell^2_{\bar\mu}},
\end{aligned}$$

where

$$A = \sup_{n \geqslant 0} (1+n)^{2\underline{\mu}} \sum_{k=0}^{n} (1+k)^{-2\bar{\mu}} (1+n-k)^{-2\varrho}$$

$$\leqslant \sup_{n \geqslant 0} (1+n)^{2\underline{\mu}} \sum_{k=0}^{n} (1+k)^{-2\bar{\mu}} (1+n-k)^{\min\{-2\underline{\mu},\, 2\bar{\mu}-2\underline{\mu}-1-2\varepsilon'\}}$$

$$\leqslant C \sup_{n \geqslant 0} (1+n)^{2\underline{\mu}+\max\{-2\bar{\mu},\, \min\{\cdots\},\, \min\{\cdots\}-2\bar{\mu}+1+2\varepsilon'\}} = C,$$

by Lemma 4.1. ∎

PROPOSITION 4.6. *Let $\varepsilon' > 0$, and let $M \subseteq \mathbb{R} \times \mathbb{R}$ and $K \subseteq \mathbb{C}$ be bounded sets. Then there is a constant C with the following properties*:

(a) *The Toeplitz operator $T(\eta_\gamma): \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ is bounded and*

$$\|T(\eta_\gamma)\|_{\mathscr{L}(\ell^2_{\bar{\mu}},\, \ell^2_{\underline{\mu}})} \leqslant C$$

for all $\gamma \in K$ and all $(\bar{\mu}, \underline{\mu}) \in M$ with $\operatorname{Re} \gamma \geqslant \varepsilon' + \max\{\underline{\mu} - \frac{1}{2}, \underline{\mu} - \bar{\mu}\}$ and $\underline{\mu} \leqslant \bar{\mu}$.

(b) *The Toeplitz operator $T(\xi_\delta): \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ is bounded and*

$$\|T(\xi_\delta)\|_{\mathscr{L}(\ell^2_{\bar{\mu}},\, \ell^2_{\underline{\mu}})} \leqslant C$$

for all $\delta \in K$ and all $(\bar{\mu}, \underline{\mu}) \in M$ with $\operatorname{Re} \delta \geqslant \varepsilon' + \max\{-\bar{\mu} - \frac{1}{2}, \underline{\mu} - \bar{\mu}\}$ and $\underline{\mu} \leqslant \bar{\mu}$.

Proof. (a) We apply Proposition 4.5(a) with $\varepsilon'/2$ instead of ε', $a^{(+)} = \eta_\gamma$ and $\varrho = \operatorname{Re} \gamma + 1/2 - \varepsilon'/2$. From Lemma 2.1(a) and from (35) we obtain that

$$|[\eta_\gamma]_n| \leqslant C_1 (1+n)^{-1-\operatorname{Re} \gamma},$$

from which the assertion follows easily. Assertion (b) can be proved similarly. ∎

PROPOSITION 4.7. *Let $\varepsilon' > 0$, let $M \subseteq \mathbb{R} \times \mathbb{R}$ a bounded set, and let K be a compact subset of $\{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C}: \gamma + \delta \notin \mathbb{Z}^-\}$. Then there is a constant C such that the Toeplitz operator $T(\xi_\delta \eta_\gamma): \ell^2_{\bar{\mu}} \to \ell^2_{\underline{\mu}}$ is bounded and*

$$\|T(\xi_\delta \eta_\gamma)\|_{\mathscr{L}(\ell^2_{\bar{\mu}},\, \ell^2_{\underline{\mu}})} \leqslant C$$

for all $(\gamma, \delta) \in K$ and all $(\bar{\mu}, \underline{\mu}) \in M$ with $\operatorname{Re}(\gamma + \delta) \geqslant \varepsilon' + \max\{\underline{\mu} - \frac{1}{2}, -\bar{\mu} - \frac{1}{2}, \underline{\mu} - \bar{\mu}\}$ and $\underline{\mu} \leqslant \bar{\mu}$.

Proof. From Lemma 2.3 we obtain that for all $n \in \mathbb{Z}$

$$|[\xi_\delta \eta_\gamma]_n| \leqslant C_1(1+|n|\ )^{-1-\operatorname{Re}(\gamma+\delta)}.$$

We write $T(\xi_\delta \eta_\gamma)$ as the sum of a lower and an upper triangular Toeplitz matrix, and then we apply Proposition 4.5 in an appropriate manner. ∎

Finally, we establish conditions for the diagonal matrix operator M_α and its inverse M_α^{-1} to be bounded. Note that M_α is invertible if and only if $\alpha \notin \mathbb{Z}^-$.

PROPOSITION 4.8.

(a) *Let $K \subseteq \mathbb{C}$ be a compact set. Then there is a constant C such that $M_\alpha : \ell^2_\mu \to \ell^2_{\mu - \operatorname{Re}\alpha}$ is bounded and*

$$\|M_\alpha\|_{\mathscr{L}(\ell^2_\mu, \ell^2_{\mu-\operatorname{Re}\alpha})} \leqslant C$$

for all $\alpha \in K$ and all $\mu \in \mathbb{R}$.

(b) *Let $K \subseteq \mathbb{C} \backslash \mathbb{Z}^-$ be a compact set. Then there is a constant C such that $M_\alpha^{-1} : \ell^2_{\mu-\operatorname{Re}\alpha} \to \ell^2_\mu$ is bounded and*

$$\|M_\alpha^{-1}\|_{\mathscr{L}(\ell^2_{\mu-\operatorname{Re}\alpha}, \ell^2_\mu)} \leqslant C$$

for all $\alpha \in K$ and all $\mu \in \mathbb{R}$.

Proof. This follows from Lemma 2.1. Observe that

$$|(\mu_n^{(\alpha)})^{-1}| \leqslant C_1 \Gamma(1+\alpha)(1+n)^{-\operatorname{Re}\alpha}$$

for all $n \in \mathbb{N}$. ∎

5. PROOF OF THE MAIN RESULT

In this section we prove Theorem 2.5. For this purpose, we need the following theorem.

THEOREM 5.1. *Let $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon = \varepsilon_1 + \varepsilon_2 < 1$. Suppose further that* $\min\{\varrho_1, \varrho_2\} > 3/2 - \varepsilon$, *and that $b \in F\ell^{2,2}_{\varrho_1, \varrho_2}$ is an invertible function with winding number equal to zero. Moreover, let U' be the set*

$$U' = \{(\gamma, \delta) \in \mathbb{C} \times \mathbb{C} : \gamma, \delta, \gamma+\delta \notin \mathbb{Z}^-, \varrho_1 > c_1(\gamma, \delta),$$

$$\varrho_2 > c_2(\gamma, \delta), \varrho_1 + \varrho_2 > c_3(\gamma, \delta)\},$$

where c_1, c_2 and c_3 are defined as in Theorem 2.5. Then for all $(\gamma, \delta) \in U'$ we have

$$\frac{D_n(b\xi_\delta\eta_\gamma)}{G[b]^n\, D_n(\xi_\delta\eta_\gamma)} = \det(P_n + P_n A P_n),$$

where $A \in \mathscr{C}_1(\ell^2_{-1/2+\varepsilon_2-\operatorname{Re}\gamma}, \ell^2_{1/2-\varepsilon_1-\operatorname{Re}\gamma})$ is the trace class operator defined by

$$A = (A_1 + A_2 + A_3)\, M^{-1}_{\gamma+\delta}, \tag{50}$$

$$A_1 = \Gamma_{\gamma,\delta}\, T(\eta_{-\gamma})\, M_\delta\, T(b_+^{-1})\, H(b_+)\, T(\xi_\gamma\eta_\delta) \times H(\tilde{b}_-)\, T(b_-^{-1})\, M_\gamma\, T(\xi_{-\delta}), \tag{51}$$

$$A_2 = \gamma M_{\gamma+\delta}\, H(\tau_\delta)\, M_{-\gamma} H(\tilde{b}_-)\, T(b_-^{-1})\, M_\gamma\, T(\xi_{-\delta}), \tag{52}$$

$$A_3 = \delta T(\eta_{-\gamma})\, M_\delta\, T(b_+^{-1})\, H(b_+)\, M_{-\delta}\, H(\tau_\gamma)\, M_{\gamma+\delta}. \tag{53}$$

Moreover, on compact subsets of U', the trace class norm of A is uniformly bounded.

Proof. In order to show that A is a trace class operator with uniformly bounded trace norm on compact subsets of U', we choose weighted spaces ℓ^2_μ in such a way that the operators which occur in (50)–(53) become bounded operators on these spaces with uniformly bounded norm on compact subsets. The Hankel operators are considered with respect to the Hilbert–Schmidt norm. Since there appear two Hankel operators in each product of (51), (52), and (53), the operators A_1, A_2, and A_3 are uniformly bounded in the trace class norm on compact subsets. Hence so is A.

Let K be a compact subset of U'. Then there exists an $\varepsilon' > 0$ such that

$$\min\{\varrho_1, \varrho_2\} \geqslant 3\varepsilon' + 3/2 - \varepsilon, \tag{54}$$

$$\min\{\varepsilon_1, \varepsilon_2\} \geqslant \varepsilon', \tag{55}$$

and

$$\varrho_1 \geqslant 3\varepsilon' + c_1(\gamma, \delta), \tag{56}$$

$$\varrho_2 \geqslant 3\varepsilon' + c_2(\gamma, \delta), \tag{57}$$

$$\varrho_1 + \varrho_2 \geqslant 6\varepsilon' + c_3(\gamma, \delta) \tag{58}$$

for all $(\gamma, \delta) \in K$. We define $\bar{c}_1$, $\bar{c}_2$ and κ by

$$\bar{c}_1(\gamma, \delta) = \max\{0, 1 - \varepsilon_1 + \operatorname{Re}\delta, 1 - \varepsilon_1 + \operatorname{Re}(\delta - \gamma)\},$$

$$\bar{c}_2(\gamma, \delta) = \max\{0, 1 - \varepsilon_2 + \operatorname{Re}\gamma, 1 - \varepsilon_2 + \operatorname{Re}(\gamma - \delta)\},$$

$$\kappa(\gamma, \delta) = (\varrho_2 - \varrho_1 + \bar{c}_1(\gamma, \delta) - \bar{c}_2(\gamma, \delta))/2.$$

Since for all $(\gamma, \delta) \in \mathbb{C} \times \mathbb{C}$

$$\max\{c_3, \max\{c_1, 3/2 - \varepsilon\} + \max\{c_2, 3/2 - \varepsilon\}\} \geqslant \bar{c}_1 + \bar{c}_2,$$

we conclude from (54) and (56)–(58) that

$$\varrho_1 + \varrho_2 \geqslant 6\varepsilon' + \bar{c}_1(\gamma, \delta) + \bar{c}_2(\gamma, \delta)$$

for all $(\gamma, \delta) \in K$. Hence $\varrho_2 - 3\varepsilon' - \bar{c}_2 \geqslant \bar{c}_1 - \varrho_1 + 3\varepsilon'$, and it follows that for $(\gamma, \delta) \in K$

$$\varrho_2 - 3\varepsilon' - \bar{c}_2(\gamma, \delta) \geqslant \kappa(\gamma, \delta) \geqslant \bar{c}_1(\gamma, \delta) - \varrho_1 + 3\varepsilon'.$$

From this and from (54), (56), and (57) we obtain

$$\begin{aligned} \varrho_1 &\geqslant 3\varepsilon' + \max\{c_1, 3/2 - \varepsilon, \bar{c}_1 - \kappa\} \\ \varrho_2 &\geqslant 3\varepsilon' + \max\{\mathrm{c}_2, 3/2 - \varepsilon, \bar{c}_2 + \kappa\} \end{aligned} \tag{59}$$

for $(\gamma, \delta) \in K$. Now we define λ_1, λ_2, μ_1, and μ_2 by

$$\begin{aligned} \lambda_1(\gamma, \delta) &= \max\{1/2 - \varepsilon_1 + \operatorname{Re} \delta, 1/2 - \varepsilon_1 + \operatorname{Re}(\delta - \gamma)\} + \varepsilon', \\ \lambda_2(\gamma, \delta) &= \min\{-1/2 + \varepsilon_2 - \operatorname{Re} \gamma, -1/2 + \varepsilon_2 + \operatorname{Re}(\delta - \gamma)\} - \varepsilon', \end{aligned} \tag{60}$$

$$\begin{aligned} \mu_1(\gamma, \delta) &= \min\{\kappa(\gamma, \delta), \operatorname{Re}(\gamma + \delta) + 1/2, -1/2 + \varepsilon_2 + \operatorname{Re} \delta\} - \varepsilon', \\ \mu_2(\gamma, \delta) &= \max\{\kappa(\gamma, \delta), -\operatorname{Re}(\gamma + \delta) - 1/2, 1/2 - \varepsilon_1 - \operatorname{Re} \gamma\} + \varepsilon'. \end{aligned} \tag{61}$$

From (61) and from (55) it follows that

$$\operatorname{Re} \gamma \geqslant \varepsilon' + \max\{\mu_1 - \operatorname{Re} \delta - 1/2, \operatorname{Re} \gamma - \varepsilon_2, \mu_1 + \operatorname{Re}(\gamma - \delta) - \varepsilon_2 + 1/2\},$$

$$\operatorname{Re} \delta \geqslant \varepsilon' + \max\{-\mu_2 - \operatorname{Re} \gamma - 1/2, \operatorname{Re} \delta - \varepsilon_1, -\mu_2 + \operatorname{Re}(\delta - \gamma) - \varepsilon_1 + 1/2\},$$

and this implies by Proposition 4.3 and 4.8 that the operators

$$\begin{aligned} &M_{-\delta} H(\tau_\gamma) M_{\gamma+\delta} \colon \ell^2_{-1/2 + \varepsilon_2 + \operatorname{Re} \delta} \to \ell^2_{-1/2 + \varepsilon_2 - \operatorname{Re} \gamma} \to \ell^2_{\mu_1 - \operatorname{Re} \delta} \to \ell^2_{\mu_1}, \\ &M_{\gamma+\delta} H(\tau_\delta) M_{-\gamma} \colon \ell^2_{\mu_2} \to \ell^2_{\mu_2 + \operatorname{Re} \gamma} \to \ell^2_{1/2 - \varepsilon_1 + \operatorname{Re} \delta} \to \ell^2_{1/2 - \varepsilon_1 - \operatorname{Re} \gamma}, \end{aligned}$$

are Hilbert–Schmidt and that their Hilbert–Schmidt norm is uniformly bounded on K. From (60) and (55) we obtain

$$\begin{aligned} -\operatorname{Re} \gamma &\geqslant \varepsilon' + \max\{-\varepsilon_1 - \operatorname{Re} \gamma, 1/2 - \varepsilon_1 + \operatorname{Re}(\delta - \gamma) - \lambda_1\}, \\ \lambda_1 - \operatorname{Re} \delta &\geqslant 1/2 - \varepsilon_1 - \operatorname{Re} \gamma, \\ -\operatorname{Re} \delta &\geqslant \varepsilon' + \max\{-\varepsilon_2 - \operatorname{Re} \delta, 1/2 - \varepsilon_2 + \operatorname{Re}(\gamma - \delta) + \lambda_2\}, \\ \lambda_2 + \operatorname{Re} \gamma &\leqslant -1/2 + \varepsilon_2 + \operatorname{Re} \delta, \end{aligned}$$

and we conclude from Proposition 4.6 and 4.8 that the operators

$$T(\eta_{-\gamma})\,M_\delta\colon \ell^2_{\lambda_1}\to\ell_{\lambda_1-\operatorname{Re}\delta}\to\ell_{1/2-\varepsilon_1-\operatorname{Re}\gamma},$$
$$M_\gamma\,T(\xi_{-\delta})\colon \ell^2_{-1/2+\varepsilon_2+\operatorname{Re}\delta}\to\ell_{\lambda_2+\operatorname{Re}\gamma}\to\ell_{\lambda_2},$$

are bounded operators with uniformly bounded norm on K. Again from (61) it follows that $\mu_2\geqslant\mu_1$ and that

$$\operatorname{Re}(\gamma+\delta)\geqslant\varepsilon'+\max\{\mu_1-1/2,\ -\mu_2-1/2,\mu_1-\mu_2\}. \tag{62}$$

Hence, by Proposition 4.7, the norm of

$$T(\xi_\gamma\eta_\delta)\colon \ell^2_{\mu_2}\to\ell^2_{\mu_1} \tag{63}$$

is uniformly bounded on K. Further, taking into account the definitions of c_1, c_2, $\bar{c}_1$, and $\bar{c}_2$, we conclude from (59), (60), (61), and (54) the inequalities

$$\varrho_1\geqslant\max\{\lambda_1,\ -\mu_1,\lambda_1-\mu_1+1/2+\varepsilon',\ 1/2+\varepsilon'\},$$
$$\varrho_2\geqslant\max\{-\lambda_2,\mu_2,\mu_2-\lambda_2+1/2+\varepsilon',\ 1/2+\varepsilon'\}.$$

Applying Proposition 4.2 and 4.5, it follows that the operators

$$\begin{aligned} &T(b_+^{-1})\,H(b_+)\colon \ell_{\mu_1}\to\ell_{\lambda_1}\to\ell_{\lambda_1}\\ &H(\tilde{b}_-)\,T(b_-^{-1})\colon \ell_{\lambda_1}\to\ell_{\lambda_2}\to\ell_{\mu_2}\end{aligned} \tag{64}$$

are Hilbert–Schmidt with uniformly bounded Hilbert–Schmidt norm on K.

From these considerations we conclude easily that A_1, A_2, and A_3 are trace class operators acting on the spaces

$$A_1, A_2, A_3\colon \ell_{-1/2+\varepsilon_2+\operatorname{Re}\delta}\to\ell_{1/2-\varepsilon_1-\operatorname{Re}\gamma}$$

and that their trace class norm is uniformly bounded. Finally, we observe that the norm of

$$M^{-1}_{\gamma+\delta}\colon \ell_{-1/2+\varepsilon_2-\operatorname{Re}\gamma}\to\ell_{-1/2+\varepsilon_2+\operatorname{Re}\delta}$$

is uniformly bounded by Proposition 4.8. This proves the desired assertion for A.

Now we are going to prove the identity (49). From (11) we obtain

$$\begin{aligned} T(b\xi_\delta\eta_\gamma)\,G[b]^{-1}&=T(b_+b_-\xi_\delta\eta_\gamma),\\ D_n(b\xi_\delta\eta_\gamma)\,G[b]^{-n}&=D_n(b_+b_-\xi_\delta\eta_\gamma).\end{aligned} \tag{65}$$

Further, we have

$$T(b_+ b_- \xi_\delta \eta_\gamma) = T(b_+)\, T(\xi_\delta \eta_\gamma)\, T(b_-) + H(b_+)\, T(\xi_\gamma \eta_\delta)\, H(\tilde{b}_-)$$
$$+ T(b_+)\, H(\xi_\delta \eta_\gamma)\, H(\tilde{b}_-) + H(b_+)\, H(\xi_\gamma \eta_\delta)\, T(b_-) \qquad (66)$$

For $\mathrm{Re}(\gamma+\delta) > -1$, this is an immediate consequence of the well-known formulas

$$T(ab) = T(a)\, T(b) + H(a)\, H(\tilde{b}),$$
$$H(ab) = T(a)\, H(b) + H(a)\, T(\tilde{b}),$$

where $\tilde{b}$ is the function $\tilde{b}(t) = b(1/t)$, $t \in \mathbb{T}$. However, since we consider $\xi_\delta \eta_\gamma$ as a distributions, some more care is in order. First we remark that the expression on the right hand side of (66) is a well defined infinite matrix C, since $T(b_+)$ and $T(b_-)$ are lower (resp. upper) triangular matrices and since

$$T(\xi_\gamma \eta_\delta),\, H(\xi_\gamma \eta_\delta),\, H(\xi_\delta \eta_\gamma) : \ell^2_{\mu_2} \to \ell^2_{\mu_1}$$
$$H(b_+) : \ell^2_{\mu_1} \to \ell^2_{\lambda_1},$$
$$H(\tilde{b}_-) : \ell^2_{\lambda_2} \to \ell^2_{\mu_2},$$

are bounded operators (see (63) and (64), and combine (62) with Proposition 4.4). By the definition of Toeplitz and Hankel operators, the (i, j)-entry of C is equal to

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} [b_+]_{i-k}\, [\xi_\delta \eta_\gamma]_{k-l}\, [b_-]_{l-j}. \qquad (67)$$

Since $b_+, b_- \in F\ell^{2,2}_{\varrho_1, \varrho_2}$ and $\min\{\varrho_1, \varrho_2\} > -1/2 - \mathrm{Re}(\gamma+\delta)$, the remarks following Proposition 2.4 apply, and formula (27) (with $b = b_+$ and $c = b_-$) is valid. Hence (67) is equal to $[(b_+ b_-)\, \xi_\delta \eta_\gamma]_{i-j}$. But this means that $T(b_+ b_- \xi_\delta \eta_\gamma) = C$, and (66) is proved.

Multiplying (66) from the left and right with $T(b_+^{-1})$ and $T(b_-^{-1})$, respectively, yields

$$T(b_+^{-1})\, T(b_+ b_- \xi_\delta \eta_\gamma)\, T(b_-^{-1}) = T(\xi_\delta \eta_\gamma) + B,$$

where

$$B = T(b_+^{-1})\, H(b_+)\, T(\xi_\gamma \eta_\delta)\, H(\tilde{b}_-)\, T(b_-^{-1})$$
$$+ H(\xi_\delta \eta_\gamma)\, H(\tilde{b}_-)\, T(b_-^{-1}) + T(b_+^{-1})\, H(b_+)\, H(\xi_\gamma \eta_\delta).$$

Because $P_n T(b_+^{-1}) = T_n(b_+^{-1})$, $T(b_-^{-1}) P_n = T_n(b_-^{-1})$ and $D_n(b_+^{-1}) = D_n(b_-^{-1}) = 1$, it follows that

$$
\begin{aligned}
D_n(b_+ b_- \xi_\delta \eta_\gamma) &= \det(T_n(\xi_\delta \eta_\gamma) + P_n B P_n) \\
&= \det(P_n + P_n B P_n T_n^{-1}(\xi_\delta \eta_\gamma)) \cdot D_n(\xi_\delta \eta_\gamma). \qquad (68)
\end{aligned}
$$

From formula (43) we obtain

$$
P_n B P_n T_n^{-1}(\xi_\delta \eta_\gamma) = \Gamma_{\gamma,\delta} P_n B P_n M_{\gamma,n} T_n(\xi_{-\delta}) M_{\gamma+\delta,n}^{-1} T_n(\eta_{-\gamma}) M_{\delta,n}.
$$

Further, from the well-known fact that $\det(e+ab) = \det(e+ba)$, we conclude that

$$
\begin{aligned}
&\det(P_n + P_n B P_n T_n^{-1}(\xi_\delta \eta_\gamma)) \\
&\quad = \det(P_n + \Gamma_{\gamma,\delta} T_n(\eta_{-\gamma}) M_{\delta,n} P_n B P_n M_{\gamma,n} T_n(\xi_{-\delta}) M_{\gamma+\delta,n}^{-1}) \\
&\quad = \det(P_n + P_n A P_n). \qquad (69)
\end{aligned}
$$

The latter conclusion is immediate from

$$
\begin{aligned}
P_n A_1 M_{\gamma+\delta}^{-1} P_n &= \Gamma_{\gamma,\delta} T_n(\eta_{-\gamma}) M_{\delta,n} P_n T(b_+^{-1}) H(b_+) T(\xi_\gamma \eta_\delta) H(\tilde{b}_-) \\
&\qquad \times T(b_-^{-1}) P_n M_{\gamma,n} T_n(\xi_{-\delta}) M_{\gamma+\delta,n}^{-1}, \\
P_n A_2 M_{\gamma+\delta}^{-1} P_n &= \Gamma_{\gamma,\delta} T_n(\eta_{-\gamma}) M_{\delta,n} P_n H(\xi_\delta \eta_\gamma) H(\tilde{b}_-) T(b_-^{-1}) \\
&\qquad \times P_n M_{\gamma,n} T_n(\xi_{-\delta}) M_{\gamma+\delta,n}^{-1}, \\
P_n A_3 M_{\gamma+\delta}^{-1} P_n &= \Gamma_{\gamma,\delta} T_n(\eta_{-\gamma}) M_{\delta,n} P_n T(b_+^{-1}) H(b_+) H(\xi_\gamma \eta_\delta) \\
&\qquad \times P_n M_{\gamma,n} T_n(\xi_{-\delta}) M_{\gamma+\delta,n}^{-1},
\end{aligned}
$$

where these identities follow from (47) and (48) and the fact that $T(\eta_{-\gamma})$ and $T(\xi_{-\delta})$ are lower (resp. upper) triangular matrices. Combining (65), (68), and (69) proves (49). ∎

For the proof of Theorem 2.5, we need also the following relation about determinants, which sharpens a result of Simon [14].

PROPOSITION 5.2. *Let H be a separable Hilbert space, and let $A, B \in \mathscr{C}_1(H)$. Then*

$$
|\det(I+A) - \det(I+B)| \leqslant \|A-B\|_{\mathscr{C}_1(H)} \exp(\max\{\|A\|_{\mathscr{C}_1(H)}, \|B\|_{\mathscr{C}_1(H)}\}).
$$

We consider the Hilbert spaces $H_1 = \ell^2_{1/2-\varepsilon_1-\operatorname{Re}\gamma}$ and $H_2 = \ell^2_{-1/2+\varepsilon_2-\operatorname{Re}\gamma}$. As it is stated in Theorem 5.1, the operator $A\colon H_1 \to H_2$ defined in (50) is a trace class operator and the corresponding norm is uniformly bounded on compact subsets of U'. Since H_1 is continuously embedded in H_2, i.e., $I \in \mathscr{L}(H_1, H_2)$, both A and AP_n are trace class operators on H_1, and

$$\|AP_n\|_{\mathscr{C}_1(H_1)} \leqslant \|A\|_{\mathscr{C}_1(H_1)} \leqslant \|A\|_{\mathscr{C}_1(H_2, H_1)}.$$

Proposition 5.2 implies now that

$$|\det(I+A) - \det(I+AP_n)| \leqslant \|AQ_n\|_{\mathscr{C}_1(H_1)} \exp(\|A\|_{\mathscr{C}_1(H_2, H_1)}).$$

and, since $\|Q_n\|_{\mathscr{L}(H_1, H_2)} = (1+n)^{-1+\varepsilon}$, we obtain

$$\|AQ_n\|_{\mathscr{C}_1(H_1)} \leqslant \|A\|_{\mathscr{C}_1(H_2, H_1)} \|Q_n\|_{\mathscr{L}(H_1, H_2)} = \|A\|_{\mathscr{C}_1(H_2, H_1)}(1+n)^{-1+\varepsilon}.$$

Finally, the determinant $\det(P_n + P_n A P_n)$ is equal to

$$\det(P_n + P_n A P_n) = \det(I + P_n A P_n) = \det(I + AP_n).$$

Combining these inequalities, we arrive at

$$|\det(P_n + P_n A P_n) - \det(I+A)| \leqslant (1+n)^{-1+\varepsilon} \|A\|_1 \exp(\|A\|_1),$$

where the norm $\|A\|_1 := \|A\|_{\mathscr{C}_1(H_2, H_1)}$ is uniformly bounded on compact subsets of U'.

Formula (49) in Theorem 5.1, involves that

$$\left| \frac{D_n(b\xi_\delta \eta_\gamma)}{G[b]^n D_n(\xi_\delta \eta_\gamma)} - \det(I+A) \right| \leqslant C_1(1+n)^{-1+\varepsilon}.$$

and, after applying Corollary 3.2, we obtain

$$\left| \frac{D_n(b\xi_\delta \eta_\gamma)}{G[b]^n (1+n)^{\gamma\delta}} - \frac{G(1+\gamma)\, G(1+\delta)}{G(1+\gamma+\delta)} \det(I+A) \right| \leqslant C_2(1+n)^{-1+\varepsilon}. \tag{70}$$

Clearly, this inequality holds for values (γ, δ) of a compact subset K of U', and the constant C_2 depends on K, ε and b only.

By the analyticity of the Fourier coefficients of $b\xi_\delta \eta_\gamma$ (see Proposition 2.4(a) and the discussion afterwards), the expressions

$$\frac{D_n(b\xi_\delta \eta_\gamma)}{G[b]^n (1+n)^{\gamma\delta}} \tag{71}$$

are analytic functions in γ and δ on U. Formula (70) involves that these functions converge uniformly on compact subset K of U' to

$$E' = \frac{G(1+\gamma)\,G(1+\delta)}{G(1+\gamma+\delta)}\det(I+A),$$

which is therefore analytic on U', too. For particular values of γ and δ (e.g. if $\operatorname{Re}\gamma \geqslant 0$ and $\operatorname{Re}\delta \geqslant 0$, see [8]), it has already been shown that $E' = E$, where E is the constant defined in Theorem 2.5. Hence, by analyticity, $E' = E$ on all of U'. (This can be verified also by a direct calculation of $\det(I+A)$, which is however a bit troublesome and is therefore omitted. Nevertheless this calculation is illuminating since it illustrates why the single terms in the product of (31) appear.)

We have proved that for each compact subset K of U' there is a constant C such that

$$\left| \frac{D_n(b\xi_\delta\eta_\gamma)}{G[b]^n(1+n)^{\gamma\delta}} - E \right| \leqslant C(1+n)^{-1+\varepsilon}$$

for all $(\gamma, \delta) \in K$ and all $n \in \mathbb{N}$. By the maximum modulus principle for analytic functions, we obtain the same assertion for U instead of U'. For this observe that (71) as well as E are analytic functions in γ and δ on U. This completes the proof of Theorem 2.5.

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